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**The Rationality Debate:
Application of Cognitive Psychology to Mathematics Education**

Uri Leron and Orit Hazzan

Department of Science Education

Technion – Israel Institute of Technology

Haifa 32000, ISRAEL

uril@tx.technion.ac.il, oritha@tx.technion.ac.il

Tel: (9724) 8293 113 Fax: (9724) 829 5634

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ABSTRACT. *Research in mathematics education usually attempts to look into students' learning and other mental processes. It could therefore be expected to build on knowledge acquired within the academic discipline of cognitive psychology. Our aim in this paper is to show how some recent developments in cognitive psychology can help interpret empirical results from mathematics education. In particular, we will be looking into the heuristics-and-biases research by Kahneman & Tversky, the alternative views by Gigerenzer et al., and the more recent dual-process theory that has come to play a central role in interpreting this research. We first introduce the relevant background from cognitive psychology and survey its connections to previous work in mathematics education; then we apply this theoretical framework for re-interpreting previously-published empirical data from mathematics education research. We conclude with a discussion of potential theoretical and practical benefits of such synthesis.*

Key words: cognitive psychology, heuristics and biases, dual-process theory, misconceptions, rationality debate.

Running head: The Rationality Debate and Math Education

The Rationality Debate: Application of Cognitive Psychology to Mathematics Education

1. INTRODUCTION

A few years ago we published in this journal an article called *The world according to Johnny: A coping perspective in mathematics education* (Leron & Hazzan, 1997), in which we presented the idea that students in mathematical problem-solving situations often experience confusion and loss of meaning. In these situations, affective and social factors are as much part of the student's thinking and behavior as are cognitive factors. These additional factors might include, for example, the need to make sense and the need to meet expectations of the authority figure involved (e.g., teacher or researcher). In that paper we attempted to analyze students' responses, taking into account such additional factors. A closely related analysis was carried out independently at about the same time by Vinner (1997), who discusses “pseudo-conceptual” and “pseudo-analytical” thought processes in mathematical learning and problem solving.

Recently we have found that what we raised as an intuitive theoretical framework has solid theoretical and empirical backing from research in cognitive psychology, mainly the heuristics-and-biases research by Kahneman & Tversky, and the alternative views by Gigerenzer (Gilovich, Griffin, & Kahneman, 2002; Kahneman, 2002; Gigerenzer & Todd, 1999; Todd & Gigerenzer, 2000; Stanovich & West, 2000, 2003). This situation has not been atypical. Since research in mathematics education usually attempts to look into students' learning and other mental processes, it could be expected to build on knowledge acquired within the academic discipline of cognitive psychology. Our aim in this paper is to show how some recent developments in cognitive psychology can help interpret empirical results from mathematics education. Conversely, psychologists may be interested, perhaps even surprised, to learn that some of their findings about people's “non-normative” behavior in mundane everyday situations are also manifest in situations of advanced mathematical tasks. In particular, we will be looking into the heuristics-and-biases research by Kahneman & Tversky, the alternative views by Gigerenzer et al., and the more recent dual-process theory that has come to play a central role in interpreting this research.

We will first introduce the relevant background from cognitive psychology and compare it with related models from mathematics education research (Section 2), and then use this theoretical framework to interpret previously-published empirical data from mathematics education research (Section 3). In conclusion, we highlight some theoretical and practical implications of this synthesis (Section 4).

2. THEORETICAL BACKGROUND

2.1 A Brief Historical Survey

The second half of the 20th century saw many age-old philosophical debates becoming the subject of empirical research. Examples of such debates are the existence of human nature vs. the *tabula rasa* assumption (e.g., Pinker, 2002); nature vs. nurture (e.g., Ridley, 2003); and whether humans essentially are or are not rational beings (e.g., Gilovich, Griffin, & Kahneman, 2002; Gigerenzer & Todd, 1999; Samuels, Stich & Tremoulet, 1999). We note in passing that the “solution” to these grand debates is typically not found in the victory of one side over the other, but in some modern synthesis that shows that the real problem was the actual *formulation* of the debates, and the inappropriate *categorization* of the world that this formulation had tacitly assumed. Thus it is now widely accepted that we are not born *tabula rasa*, that is, human nature exists, but also that our nature develops *via* nurture; and that there is not one but several kinds of rationality.

The background to this paper is the so-called *rationality debate*: Are humans essentially rational beings or not? Or, better, how rational are human beings? Or, still better, what kind of rationality (or irrationality) is invoked under what conditions?

This question had been endlessly debated by the great philosophers through the millennia, was completely suppressed by behaviorist psychology in the first half of the 20th century, and has been taken up centrally and empirically by cognitive psychologists (notably Kahneman and Tversky) in the second half of the 20th century.

Initially, the extensive empirical work by Kahneman and Tversky was mainly interpreted as demonstrating that under many conditions – mainly involving decision under uncertainty and statistical thinking – people behave irrationally: They would give answers that contradicted “normative” (i.e., mathematical or statistical) answers and, moreover, gave contradictory answers under different but logically equivalent

formulations of the tasks (the so-called “framing effects”). This research gave rise to an extensive literature in cognitive psychology, generically termed the *heuristics-and-biases* (H&B) program, and summarized in Kahneman, Slovic & Tversky (1982) and, more recently, in Gilovich, Griffin, & Kahneman (2002).

This irrationality interpretation – though mostly not the empirical findings themselves – was challenged by various other psychologists, who used different definitions of rationality to argue that people are much more rational than would appear from the H&B literature. Such challenges came, for example, from evolutionary psychology and from researchers like Gigerenzer, who claimed that people’s behavior is basically rational if one takes into account our evolutionary heritage, realistic conditions of the environment, the limited information under which decisions are usually made, and the limitations of our brain’s computational capabilities. The terms *bounded rationality* (Simon, 1982) and *ecological rationality* (Gigerenzer & Todd, 1999) were coined to stress the limited resources and the role of the environment in our cognition and decision making.

In the last 20 years, much of the “negative interpretation” of the H&B research has been softened to accommodate some of the criticism of the ecological rationality camp, and even (more hesitantly) the evolutionary psychology one. The more balanced attitude between the “negative” and “positive” interpretations is presented in Gilovich, Griffin, & Kahneman (2002). This research has found applications in several disciplines, and in 2002 led to Kahneman being awarded the Nobel Prize in economy (Tversky unfortunately having in the meantime passed away).

Differences still remain, but for the applications to mathematics education research that will be presented here, they are of little significance, and we will draw freely from the various interpretations of H&B literature. Following Kahneman (2002), our focus will be on the new theoretical framework in which the H&B research has been recast, namely *dual-process* theory, which we next survey.

2.2 Dual-Process Theory

A large part of mathematics education research is concerned (explicitly or implicitly) with the relationship between the intuitive and analytical modes of thinking and behavior (e.g., Fischbein, 1987; Stavy and Tirosh, 2000). Empirical findings on

misconceptions are often explained by the mismatch between the students' intuitions and the requirements of the formal system of mathematics. As it turns out, the distinction between intuitive and analytical modes of thinking is rigorously and solidly expressed in what cognitive psychologists call *dual-process theory*. In fact there are several such theories but since the differences are of little significance in this first application to mathematics education, we will ignore here the nuances, and will adopt the generic framework presented in Stanovich & West (2000), Gilovich, Griffin, & Kahneman (2002), and Kahneman (2002).

According to this theory, our cognition and behavior operate in parallel in two quite different modes, called *System 1* (S1) and *System 2* (S2), roughly corresponding to our common sense notions of intuitive and analytical thinking. These modes operate in different ways, are activated by different parts of the brain, and have different evolutionary origins (S2 being evolutionarily more recent and, in fact, largely reflecting *cultural* evolution). The distinction between perception and cognition is ancient and well known, but the introduction of S1, which sits halfway between perception and (analytical) cognition is relatively new, and has important consequences for how empirical findings in cognitive psychology are interpreted, including the rationality debate and the application to mathematics education research.

Like perception, S1 processes are characterized as being fast, automatic, effortless, unconscious and inflexible (hard to change or overcome); unlike perception, S1 processes can be language-mediated and relate to events not in the here-and-now (i.e., events in far-away locations and in the past or future). In contrast, S2 processes are slow, conscious, effortful and relatively flexible. The two systems differ mainly on the dimension of *accessibility*: how fast and how easily things come to mind. In many situations, S1 and S2 work in concert, but there are situations (such as the ones concocted in the H&B research) in which S1 produces quick automatic non-normative responses, while S2 may or may not intervene in its role as monitor and critic. The various distinctions concerning the two systems are summarized in Figure 1 (taken from Kahneman, 2002, p. 451).

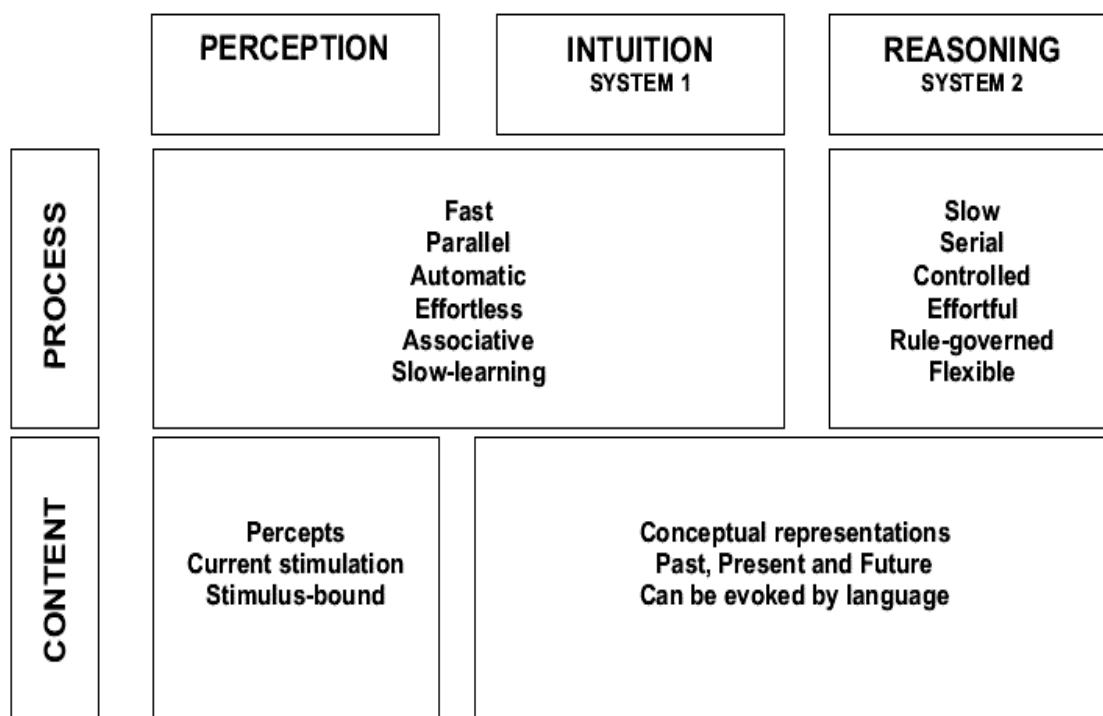


Figure 1. Properties of System 1 and System 2 (Kahneman, 2002)

Many of the non-normative answers people give in psychological experiments – and in mathematics education tasks, for that matter – can be explained by the quick and automatic responses of S1 and the frequent failure of S2 to intervene in its role as critic of S1. Or, put in Kahneman’s (2002) words:

Absent a system that reliably generates appropriate canonical representations, intuitive decisions will be shaped by the factors that determine the accessibility of different features of the situation. Highly accessible features will influence decisions, while features of low accessibility will be largely ignored. Unfortunately, there is no reason to believe that the most accessible features are also the most relevant to a good decision. (p. 459)

Here is a striking example (taken from Kahneman, 2002) for the tendency of the fast-reacting S1 to “hijack” the subject’s attention and lead to non-normative answers.

A baseball bat and ball cost together one dollar and 10 cents. The bat costs one dollar more than the ball. How much does the ball cost?

Almost everyone reports an initial tendency to answer ‘10 cents’ because the sum \$1.10 separates naturally into \$1 and 10 cents, and 10 cents is about the right magnitude. Frederick found that many intelligent people yield to this immediate impulse: 50% (47/93) of Princeton students, and 56% (164/293) of students at the University of Michigan gave the wrong answer. (p. 451)

According to dual-process theory, this situation presents “cognitive illusions”, analogous to the famous optical illusions known from cognitive psychology. The salient features of the problem cause S1 to jump immediately with the answer of 10 cents, since the numbers one dollar and 10 cents are salient, and since the orders of magnitude are roughly appropriate. Many people accepted S1’s response uncritically, thus in a sense “behaved irrationally”. For others, in contrast, S1 had also immediately jumped with this answer, but in the next stage, their S2 interfered critically and made the necessary adjustments to the answer. Significantly, the way S1 operated here, namely making a very quick decision based on salient features of the problem and of rough approximation, usually gives good results under natural conditions, such as searching for food or avoiding predators. Hence the insistence of Gigerenzer (e.g., Gigerenzer & Todd, 1999) that this is a case of *ecological rationality* being fooled by a tricky task, rather than a case of irrationality. The various debates arising from the interpretations of the H&B research form a fascinating topic which is, however, beyond the scope of this paper.

The S1/S2 differences are also relevant to an extremely important issue in mathematical understanding and problem solving – that of thinking in and out of context. Stanovich and West (2000) point to an intriguing connection to personal and social context:

For the purposes of the present discussion, the most important difference between the two systems is that they tend to lead to different types of task construals. Construals triggered by System 1 are highly contextualized, personalized, and socialized. [...] The primacy of these mechanisms leads to what has been termed the fundamental computational bias in human cognition [...] – the tendency toward automatic contextualization of problems. In contrast, System 2’s more controlled processes serve to decontextualize and depersonalize

problems. This system is more adept at representing in terms of rules and underlying principles. It can deal with problems without social content and is not dominated by the goal of attributing intentionality nor by the search for conversational relevance. (pp. 658-659)

It is important to note that skills can migrate between the two systems. When a person becomes an expert in some skill, perhaps after a prolonged training, it may become S1 skill for this person. For example, driving is an effortful S2 behavior for beginners, requiring deep concentration and full attention; for experienced drivers, however, driving becomes an S1 skill, which they can perform automatically while engaged in a deep intellectual or emotional conversation. On the other hand, many S1 skills (such as walking straight or conversing in a foreign language), with advancing age, or when just being tired or drunk, suddenly require conscious awareness to perform successfully.

Stanovich and West (2000) hint at the origins of the two systems by distinguishing between *evolutionary rationality*, which stems from ancient origins, and *normative rationality*, which reflects more the needs of modern, technological society:

It must be stressed though that in the vast majority of mundane situations, the evolutionary rationality embodied in System 1 processes will *also* serve the goals of normative rationality. [...]

Thus, situations where evolutionary and normative rationality dissociate might well put the two processing Systems in partial conflict with each other. These conflicts may be rare, but the few occasions on which they occur might be important ones. This is because knowledge-based, technological societies often put a premium on abstraction and decontextualization, and they sometimes require that the fundamental computational bias of human cognition be overridden by System 2 processes. (p. 661)

An additional intriguing speculation about the evolutionary origins of the two systems is found in Stanovich and West (2003). They postulate that S1 has evolved to serve the “interests” of the genes (the *Replicators* in Dawkins’ (1990/1976) terms); S2, in comparison, has evolved to serve the “interests” of the whole organism (the *vehicle* or

the *interactor*). As before, those “interests” are usually in harmony, but in some cases may be in conflict.

It is appropriate to close this summary with words from the conclusion of Kahneman’s (2002) Nobel Prize lecture:

The starting point of the present analysis was the observation that complex judgments and preferences are called ‘intuitive’ in everyday language if they come to mind quickly and effortlessly, like percepts. Another basic observation was that judgments and intentions are normally intuitive in this sense, but can be modified or overridden in a more deliberate mode of operation. The labels ‘System 1’ and ‘System 2’ were associated with these two modes of cognitive functioning.

The preceding sections elaborated a single generic proposition: “Highly accessible impressions produced by System 1 control judgments and preferences, unless modified or overridden by the deliberate operations of System 2.” This template sets an agenda for research: to understand judgment and choice we must study the determinants of high accessibility, the conditions under which System 2 will override or correct System 1, and the rules of these corrective operations. Much is known about each of the three questions. (p. 481)

2.3 Intuition vs. Analytical Thinking in Mathematics Education and in Psychology: A Comparative Summary

At this point readers might have noticed some close similarities between Dual-Process theory on the one hand, and familiar theoretical constructs from the mathematics education research literature on the other. In this section we analyze the similarities and differences between the two frameworks and speculate on their sources.

The results of the upcoming analysis are summarized in Figure 2. The rest of this section will elaborate on this Figure.

Mathematics Education Perspective	Cognition		Metacognition	
	Intuition	Analytical thinking	Self monitoring	Beliefs, Resource management, etc.
Dual-Process Perspective	System 1	Serial, rule-governed thinking	Self monitoring	
		System 2		
	Cognition			

Figure 2. A comparison of terminology between mathematics education and dual-process theory

The first similarity, as can be seen in the left column of Figure 2, is between System 1 on the one hand and Intuition on the other. Indeed, in Kahneman's (2002) diagram (reproduced in this paper as Figure 1), the headings "System 1" and "Intuition" appear together. Examination of the mathematics education literature shows that the definition and function of intuition in it are also similar to those of S1. Both are characterized by immediacy, high accessibility, automaticity and effortlessness. Also both are considered mostly useful and reliable under normal everyday conditions but are prone to errors under more complex and abstract conditions, especially due to distraction by irrelevant external clues of high accessibility.

Here, for example, are a few quotes from two important books on intuition in mathematics and science education:

Intuitive knowledge is immediate knowledge; that is, a form of cognition which seems to present itself to a person as being self-evident. [...] In all these instances, one deals with apparently immediate forms of cognition. (Fischbein, 1987, p. 6; italics in the original)

What is surprising is the fact that [the subject] is ready, in certain circumstances, to ignore completely an important fact of the relevant information for the sake of immediacy. Since the selection is automatic the subject not only does not use a part of the information,

but he even ignores its importance for the validity of the judgment.

(ibid, p. 108)

The essential claim of the intuitive rules theory is that human responses are determined mainly by irrelevant external features of the tasks, not by related concepts and ideas. (Stavy and Tirosh, 2000, p. 85)

The second similarity is between the self-monitoring function of S2 and the same part of metacognition (third column of Figure 2). However, as can be seen from the figure, both S2 and metacognition consist of more than this monitoring, or self-regulating, function, though those additional parts are different in the two frameworks. These differences will be elaborated below.

The third similarity is in the use of the term “cognition”; however, this is only a partial similarity, which is expressed in the left half of the figure. In the cognitive psychology literature, cognition encompasses all thought processes, indeed all information processing. This includes S1 (including unconscious processes) and S2 (including the monitoring component). In mathematics education the use of “cognition” varies somewhat. Most seem to include intuition within cognition, as the above quotes from Fischbein demonstrate, but compare this with the position of Vinner (1997) who seems to reserve cognition for analytical thinking only:

[...] much effort is devoted to find cognitive interpretations for many types of behavior for which, perhaps, a different type of interpretation is more suitable. Furthermore, much didactic effort is invested in ‘cognitive corrections’ where perhaps a different type of correction would be more effective. By saying this, I am not denying the importance of cognitive research. I am asserting, however, that not every event in a mathematics learning can be explained in cognitive terms, and that it is a fallacy to assume that the cognitive approach is adequate for almost every situation in mathematics learning. (p. 97-98. As the rest of his article shows, Vinner seems to reserve the term cognition for analytical thinking.)

Having outlined some central similarities between the two research communities, we now turn to discussing the differences.

We start with the difference in the use of the term “cognition”. In mathematics education, cognition is mainly used in conjunction (and in contrast) with metacognition; hence cognition does not include the monitoring function (thinking about thinking, checking your own problem-solving behavior), which appears in Figure 2 as part of metacognition. In cognitive psychology, in contrast, the monitoring function is part of S2, which is definitely part of cognition.

An additional difference that is illustrated in the figure is that in mathematics education metacognition consists of more than the self-monitoring function of S2, as described by Schoenfeld (1987):

Research on metacognition has focused on three related but distinct categories of intellectual behavior:

1. Your knowledge about your own thought processes. How accurate are you in describing your own thinking?
2. Self-awareness, or self-regulation. How well do you keep track of what you're doing when (for example) you're solving problems, and how well (if at all) do you use the input from such observations to guide your problem solving actions?
3. Beliefs and intuitions. What ideas about mathematics do you bring to your work in mathematics, and how does that shape the way you do mathematics? (p. 190)

We can summarize the above differences by describing how the two disciplines view the top-level division of the various categories. In both disciplines the top level is divided into two main categories, but the divisions are different: in mathematics education they are cognition and metacognition, while in cognitive psychology they are System 1 and System 2. Looking at the diagram vertically, we can observe the following three relations. First, cognition as used in mathematics education consists (roughly) of S1 and the analytical thinking part of S2. Second, S2 as used in cognitive psychology consists of part of cognition in mathematics education (the analytical thinking part) and part of metacognition (the self-monitoring part). Third, metacognition shares with S2 the self-monitoring part, but contains an additional functionality (beliefs, etc.).

We conclude this section with some thoughts on the *source* of the differences between the two disciplines. We propose that the difference in conceptualization stems from the difference in the *goals* of the two communities. In operational terms, it comes from the different kind of problems the two communities are interested in and, consequently, the kind of mental processes in which they expect their subjects to engage. If we look for example at the Heuristics and Biases program (Kahneman, Slovic and Tversky, 1982), we can see that their aim is to examine everyday cognition, using mostly simple and mundane tasks. Moreover, these researchers do not expect their subjects to think hard and long or to give explanations and justifications for their answers. As can be seen from the bat-and-ball task, to take a typical example, if S2 at all intervenes in its role as critic and monitor of S1, the resulting thought process involves both the reasoning and the self-monitoring functions of S2 in an inseparable fashion. S2 may notice that 10 cents is inappropriate, because 1 dollar is not 1 dollar more than 10 cents. This process involves both functions and it seems impossible to say where the critic ends and the calculation begins. This may explain why in theorizing about those experiments, the more compact S2 may have done a better explanatory job than metacognition.

In mathematics education, in contrast, the goal is to study mathematical understanding and problem-solving of students who were previously tutored in the relevant subject matter (e.g., we do not expect students to understand algebraic equations or group theory without explicit tutoring). The subjects in such research are typically expected to spend significant amount of time on the solution, to think hard, and to explain their answers. Under those conditions, as amply demonstrated by Schoenfeld (1985), metacognition in its extended scope becomes crucial: in addition to self-monitoring, factors such as belief systems (“if it cannot be solved in 10 minutes it cannot be solved at all”) and time management (not going off on a wild goose chase) are essential. In comparison, these factors have no opportunity to be expressed in the simple tasks used in the heuristic and biases research.

2.4 Applications in Mathematics Education Research

In the next section we will examine several phenomena discussed in the mathematics education research literature. Our aim is to use the insights from the rationality

debate, in particular the dual-process theory, for re-interpreting these phenomena. For example, the students-and-professors phenomenon, discussed more thoroughly below, seems analogous to the bat-and-ball one. We need to look carefully, though, at some of the differences. First, is there at all any S1 thinking in the context of mathematical problems, where the name of the game supposedly is all about rational thinking and reasoning? We think yes, because students need to give *some* answer, and will give the one that comes most easily to mind and that looks roughly appropriate; this does look like S1 behavior. Second, it is likely that many of the students who answered incorrectly on the students-and-professors task, do know how to arrive at the correct answer, and will give it if prompted or challenged. However, the S1 answer is the one that immediately jumps to their attention, and there is nothing in the situation to alert the critical intervention of S2. Or, to use Kahneman's (2002) words:

Neither of these examples of reasoning exceeds the intellectual reach of the graduate students at major universities [...]. However, the data indicate that very few respondents actually came up with corrections.
(p. 472)

The theoretical framework presented here is meant to complement existing frameworks, not to replace them. Its "added value" is in explaining persistent errors in mathematical tasks by invoking general cognitive considerations, which are not limited to mathematics. Thus this explanation of the students-and-professors errors is not concerned with knowledge of equations, or any other mathematical knowledge, but with the general cognitive process, by which S1 hijacks the students' attention too fast for S2 to act. In the standard approach, we would instead analyze the mathematical or logical difficulties the students faced, assuming there is some deficiency in their mathematical knowledge. In the dual-process framework, there is no need to assume any gaps in the relevant mathematical (S2) knowledge: the emphasis here is that this knowledge did not even have the chance to be expressed. In other words, *standard analyses assume that the answer is a buggy product of S2, while in fact it is a product of S1*. In reality, it is likely that both types of errors occur. Further empirical research is needed to determine their relative frequency.

3. CASE STUDIES

In this section we re-examine two case studies from mathematics education research, in order to show how dual-process theory can be applied to interpreting such data. One case study is from the general literature – the students-and-professors problem, and the other is from our own research – the use and misuse of mathematical theorems. In both cases we illustrate the joint action of S1 and S2 in producing the non-normative answers, as in the following quote from Kahneman (2002):

The present treatment assumes that System 2 is involved in all voluntary actions—including overt expressions of the intuitive judgments that originated in System 1. This assumption implies that errors of intuitive judgment involve failures of both systems: System 1, which generated the error, and System 2 which failed to detect and correct it [...]. (p. 471)

3.1. The Students-and-Professors Problem

The following task and data are taken from Clement, Lochhead & Monk (1981).

The task: Write an equation for the following statement: "There are six times as many students as professors at this university." Use S for the number of students and P for the number of professors. (p. 288)

The data: On a written test with 150 calculus-level [engineering] students, 37% missed this problem, and two-thirds of the errors took the form of a reversal of variables such as $6S = P$. In a sample of 47 non-science majors taking college algebra, the error rate was 57%. (p. 288)

Standard explanation: Two main explanations for this erroneous answer are offered in Clement, Lochhead & Monk (1981).

The first "faulty approach", called *word order matching*, "is a literal, direct mapping of the words of English into the symbols of algebra." (p. 288) Thus students interpret "six times as many students as professors" as *6 times students gives professors*, or $6S = P$.

Students who take the second faulty approach, called *static-comparison*, understand that “the sentence implies that the student population is much bigger than the faculty population [...], [but they still believe] that this relationship should be represented by the equation $6S = P$. Apparently the expression “ $6S$ ” is used to indicate the larger group and “ P ” to indicate the smaller group. The letter S is not understood as a variable that represents the *number* of students but rather is treated like a label or unit attached to the number 6. The equals sign expresses a comparison or association, not a precise equivalence.” (p. 288)

As the following conclusion indicates, the authors mainly attribute the error to what we would call S2's failure: “The reversal error appears to be due not simply to carelessness but rather to a self-generated, stable, and persistent misconception concerning the meaning of variables and equations.” (p. 289)

There are hints, however, of “S1 thinking” in the following quotation, as our bracketed and italicized insertions show:

We also interviewed 15 students who were asked to think out loud while solving problems like this one. The videotaped records [...] allow one to distinguish between insignificant careless [*S1*] errors and more serious conceptual [*S2*] problems. [...] In these interviews the students vacillated between correct and incorrect solutions and appeared to be thoroughly confused, not just guilty of making hasty mistakes [*they could not decide between the “voices” of S1 and S2*]. (p. 288)

Arcavi (1994) also suggests that “this error is not a manifestation of a deep misconception about the notion of variable” but rather a result of the word order acting as a distractor. This resembles S1 behavior as elaborated below.

Dual-process analysis: The analysis according to dual-process theory attributes the error to failures of *both* S1 and S2 (though, as previously mentioned, it is likely that separate S2 errors also occur). S1 invokes what is most easily accessible to the students which also looks roughly appropriate, and S2 fails in its role as critic or monitor. Specifically, we distinguish two stages – the written test and the interviews. In the written test, the students know there should be an equation, with S and P in it, one of which should be multiplied by 6. Because of the expression “6 students”, the

symbol $6S$ comes immediately to mind and thus the answer $6S = P$ is automatically and quickly invoked by $S1$. We repeat that this does not mean that they do not have the knowledge to produce the right answer, had they only stopped to think more (that is, invoke $S2$). The problem is, rather, that they have no reason to suspect that the answer is wrong, thus $S2$ remains dormant. Had $S2$ done its job properly, it would have gone through something like the following reasoning: Since there are more students than professors, we need to multiply the number of professors (P) by 6 to get the two sides equal. This explanation echoes Kahneman's (2002) words:

The surprisingly high rate of errors in this easy problem [the bat-and-ball problem] illustrates how lightly the output of System 1 is monitored by System 2: people are not accustomed to thinking hard, and are often content to trust a plausible judgment that quickly comes to mind. (pp. 451-452)

In light of the foregoing considerations, we conjecture that a small cue may be all that's necessary to lead many of the students to the right answer. In fact, the additional interviews seem to have supplied such a cue, hence they enable us to refine our analysis of the relative roles of $S1$ and $S2$ and the interaction between them. As the interviews show, $S1$ is clued by surface features, such as word order, hence pushes the student towards the " $6S = P$ " response. $S2$, in contrast, is driven by the meaning of the concepts and symbols involved, and the rules for manipulating them, hence pushes the students towards the normative " $S = 6P$ " answer. Thus some of the students "vacillate" between the messages coming from the two systems. The confidence of those students in their mathematical knowledge is apparently not strong enough to take a clear position in favor of one side.

One more experiment that seems to have awakened $S2$, was reported by Soloway, Lochhead, and Clement (1982). Instead of writing an equation, the students were asked to write a computer program that inputs the number of professors and prints the number of students. The data indicates that when the task was performed with programming, the rate of errors was reduced. A dual-process interpretation would be that the computerized version forced the students into a step-by-step sequential process in a formal language, which can only be performed by $S2$, thus blocking $S1$'s dominance.

3.2. Student's Use and Misuse of Mathematical Theorems

In this section we revisit our paper *Students' Use and Misuse of Mathematical Theorems: The Case of Lagrange's Theorem* (Hazzan and Leron, 1996). In that paper we present several observations related to how abstract algebra students (mis-)use Lagrange's theorem in different problem solving situations and suggest several explanations for them. This task deals with relatively advanced mathematics, but we make an attempt to present it here in a completely self-contained way, so that no previous knowledge of group theory is needed. This is achieved by explaining the relevant group theoretical terms only in the context of this particular example rather than in their full generality. A few generalizations and subtleties are mentioned in the footnotes and can be safely ignored. For a more thorough treatment, see Gallian (1990).

Mathematical background: The entire task is taking place within the group Z_6 , consisting of the set $\{0,1,2,3,4,5\}$ and the operation of addition modulo 6, denoted by $+_6$. Thus, for example, $2 +_6 3 = 5$, $3 +_6 3 = 0$, $3 +_6 4 = 1$, and, in general, $a +_6 b$ is defined as the remainder of the usual sum $a + b$ upon division by 6.

Z_6 is a *group* in the sense that it contains 0 and is *closed* under addition mod 6; that is, if a and b are in Z_6 , then so is $a +_6 b$.¹ Similarly, we define Z_3 to be the group consisting of the set $\{0,1,2\}$ and the operation $+_3$ of addition modulo 3. A *subgroup* of Z_6 is a subset of $\{0,1,2,3,4,5\}$ which is in itself a group under the operation defined in Z_6 . For example, it can be checked that the subset $\{0,2,4\}$ is a subgroup of Z_6 , since it contains 0 and is closed under $+_6$.

All the groups in this discussion are *finite*, in the sense that they have a finite number of elements; this number is called the *order* of the group. Thus the order of Z_6 is 6 and the order of Z_3 is 3. Finally, an important theorem of group theory, called *Lagrange's theorem*, states that *if H is a subgroup of Z_6 , then the order of H divides 6*.² Thus, for example, the order of H cannot be 4 or 5 but 3 is possible, and indeed, we have seen

¹ In the general definition of a group there are more requirements, namely associativity and the existence of "inverses", but in the present context it can be shown that these are automatically satisfied, so we need not worry about them.

² We say that an integer k *divides* an integer n if n is a whole multiple of k . For example, 3 divides 6 but 4 doesn't; 6 divides 12 but 5 doesn't.

above an example of a subgroup of Z_6 with 3 elements³. For what follows, it is relevant to mention that the *converse* of Lagrange's theorem is *not* true in general: It is possible to give an example of a group G of order 12 which does not contain a subgroup of order 6 (for details see Gallian, 1990, Example 13, p. 151).

The task and data: (Hazzan & Leron, 1996)

The following task was given to 113 computer science majors in a top-notch Israeli university, who had previously completed courses in calculus and linear algebra (an abstract approach), and were in the midst of an abstract algebra course:

A student wrote in an exam, " Z_3 is a subgroup of Z_6 ".
 In your opinion, is this statement true, partially true, or false?
 Please explain your answer.

An incorrect answer was given by 73 students, 20 of whom invoked Lagrange's theorem in essentially the following manner:

Z_3 is a subgroup of Z_6 by Lagrange's theorem, because 3 divides 6.

Mathematical remark 1. The correct answer is that Z_3 is *not* a subgroup of Z_6 . The reason is that Z_3 is not closed under the operation $+_6$ (for example, $2 +_6 2 = 4$, and 4 is not in Z_3). The question is tricky because Z_3 is a sub-*set* of Z_6 and is a group (relative to $+_3$), but it is not a sub-*group* (since it is not a group relative to $+_6$). There is a sophisticated sense in which the statement " Z_3 is a subgroup of Z_6 " is partially true, namely, that Z_3 is *isomorphic* to the subgroup $\{0, 2, 4\}$ of Z_6 . We would of course be thrilled to receive this answer, but none of our 113 subjects had chosen so to thrill us.

Mathematical remark 2. As can be seen from the previous remark, our solution does not use Lagrange's theorem. It is relevant to mention that in spite of superficial resemblance, there is no way Lagrange's theorem could even *help* on this task, since "H is a subgroup" is the hypothesis of that theorem, not its conclusion. Worse still, what the students seem to be using is some (incorrect) version of the *converse* of Lagrange's theorem.⁴

³ More generally, Lagrange's theorem applies to any two finite groups H and G : If H is a subgroup of G , then the order of H divides the order of G .

⁴ Hazzan & Leron (1996) discuss the data on two more tasks, which shows that this misuse of Lagrange's theorem is deeper and more persistent than might appear from the data presented here.

Standard explanation: Much of the interpretation offered in Hazzan & Leron (1996) falls under “standard explanation”, though some hints of what would become dual-process explanation can already be gleaned there. For the standard explanation, it suffices to give one brief sample:

In this example, students have exhibited [the following misuses of Lagrange's theorem]: failing to distinguish between Lagrange's theorem and its (naive!) converse, and in fact applying Lagrange's theorem (or its converse) where such application doesn't help in solving the given problem. (p. 25)

Looking back at our 1996 paper, we can recognize in retrospect that it already contained the seeds for our current explanation. We explained that

[this example and others like it] can be seen as special cases of two very general kinds of behaviors [*heuristics*]. One, obtaining the solution to exam problems by clinging to clues in the surface representation (the text) of the problem [*framing effects*]; two, using the theorem and its converse indistinguishably, and other converse-related bugs. [...] Looking with the student's eye, both these behaviors may be seen as representing the same state of mind: groping in the dark in an attempt to make some sense of a situation which hardly seems to be making any sense. (p. 25)

Furthermore, we specifically said that

students are clued to Lagrange's theorem by a question about subgroups, and by the appearance of two natural numbers which are the orders of groups. They then answer true or false according to whether the numbers divide or not [*S1 behavior*], without further thought, careful reading of the problem, or checking of the solution [*S2 behavior*]. (p. 25)

It is possible that these phenomena occur mainly with a certain type of theorem: perhaps one which has a name, or one which is particularly memorable for other reasons, e.g., especially simple formulation involving natural numbers. If, as in the case of Lagrange's theorem, the theorem can be memorized as a "slogan", then it can easily be retrieved from memory [*accessibility*] under the hypnotic effect of a magic incantation. However,

using a theorem as a magic incantation may increase the tendency to use it carelessly [*S1 thinking*], with no regard to the situation or to the details of its applicability [*S2 thinking*]. (p. 26)

This description fits naturally within the dual-process framework (had we been aware of it then) as presented next.

Dual-process analysis: As in the students-and-professors case, we claim that the students' answers reflect a combined S1-S2 failure.

First, the S1 response is invoked by what is most immediately accessible to the students in the situation, which also looks roughly appropriate to the task at hand; this results in the inappropriate invocation of Lagrange's theorem as described above. Specifically, the students know that using a theorem in such situations is expected; the appearance of the two numbers 3 and 6 as orders of two groups, and the fact that 3 divides 6, immediately and automatically cues Lagrange's theorem, yielding the answer, " Z_3 is a subgroup of Z_6 by Lagrange's theorem, because 3 divides 6".

Kahneman (2002) calls this process of substituting an accessible and easy situation for the given complex situation "relative strength heuristics":

[...] people who are confronted with a difficult question sometimes answer an easier one instead. [...] This could be called a "relative strength heuristic". In both cases, the target attribute is low in accessibility and another attribute, which is (i) related to the target, and (ii) highly accessible, is substituted in its place. (p. 466)

Second, S2 fails in its role as critic of S1, since there is nothing in the task situation to alert its monitoring function. The missing judgment – mainly that Lagrange's theorem cannot be used to establish the existence of a subgroup but only its absence – clearly requires S2 involvement. Incidentally, it is quite possible that some of the students do have the knowledge to produce the right answer, had they only stopped to think more (that is, invoke S2). The problem is, rather, that they have no reason to suspect that the answer is wrong, thus their S1 reaction escapes the notice of the "permissive System 2", to use Kahneman's (2002) words:

[An] evaluation of the heuristic attribute comes immediately to mind, and [...] its associative relationship with the target attribute is

sufficiently close to pass the monitoring of a permissive System 2. (p. 469)

We use the dual-process framework to further analyze the students' behavior. There are in fact *two* ways in this problem for S1 to jump ahead and lead to a non-normative answer. One (which is what we had in mind when we composed the task) is to be misled by the fact that Z_3 is both a group and a sub-*set* of Z_6 , yet it is not a sub-*group*. The other (which initially took us completely by surprise) is to be misled by the apparent relevance of Lagrange's theorem. Further experimental evidence is required to analyze the students' responses in the light of the two possible S1 influences. This shows, incidentally, that the dual-process theory leads not only to new explanations, but also (like all good theories should) to interesting new research *questions*.

In order to obtain the normative answer, the students should in fact have carried out *two* S2 processes, corresponding to the two roles of S2 in general: logical reasoning and monitoring of S1. First there is the analytical process that should have been carried out: Z_3 is $\{0,1,2\}$ with addition mod 3, Z_6 is $\{0,1,2,3,4,5\}$ with addition mode 6. Is Z_3 a subset of Z_6 ? Is it a group? What is its operation, and is it the same as that of Z_6 ? – clearly an S2 process. The second S2 process, in its role as monitor, is overriding and rejecting the S1 automatic invocation of Lagrange's theorem. Here there is need to check the conditions that are required for the application of the theorem. Such a check would indeed show that Lagrange's theorem starts by *assuming* that one group is a subgroup of another, thus cannot be used to *prove* such a claim.

4. CONCLUSION

Building bridges between the conceptualizations used in mathematics education and in cognitive psychology may benefit both communities. In this section we discuss some possible theoretical benefits (4.1) and some practical ones (Section 4.2).

4.1. Added Value

Building a bridge between the conceptualizations used by mathematics education and by cognitive psychologists has benefits for both communities. First we discuss the benefit to mathematics education.

The added value of using dual-process theory in mathematics education research is not the distinction between intuitive and analytical thinking; the distinction itself is of course ancient and well-known and much has been written about it. Rather, we see the added value in tightening, refining and operationalizing this distinction, as we now elaborate. Dual-process theory in cognitive psychology is quite solid, with vast theoretical and empirical literature to support it. (In our brief survey above we could barely touch the surface; cf. Stanovich & West (2000, 2003) for a more comprehensive survey of this literature and for references to primary sources.) There are empirical methods for determining whether a person performing a task is using S1 or S2; for example, S2 operation is very demanding of attention, therefore is more easily distracted by irrelevant questions presented to the subject.⁵ Furthermore, the H&B literature amply demonstrates the power of dual-process theory as a unified framework for explaining empirical findings on people's decision-making and problem-solving behavior (Gilovich, Griffin & Kahneman, 2002; Kahneman, 2002).

The extension of dual-process theory to mathematics education research has also added value for the psychologists themselves. As previously mentioned, the kind of tasks and the kind of expectations are very different between the two research communities. In the H&B studies, the tasks are mostly simple and universal and relate to everyday situations, which typically tend to invoke S1 responses. These tasks are usually meant to probe people's immediate, spontaneous performance, "without thinking": People are asked to make some judgment, but are not usually pressed for elaborate explanations of their answers.

The situations in mathematics education research are different in several respects. First, the mathematical tasks are typically abstract and complex and removed from everyday experience. Second, the content has been previously taught to the students in a formal setting. Third, the tasks are given in an examination or problem-solving situation, with explicit invitation for logical thinking and reasoning, which would be

⁵ It would not do to operationalize S1 and S2 by saying, e.g., that S1 consists of processes that are easy to perform, because then all our results would become vacuous: "people do more easily the things that are easy to perform". Thus, independent criteria are needed.

expected to elicit S2 behavior. Finally, the tasks are often given as part of an examination, where the students do care about the grades and are motivated to answer correctly. To illustrate, our second case study is carried out in the context of group theory, where the name of the game is definitions and theorems and proofs; and the students are not just asked whether " Z_3 is a subgroup of Z_6 ", but also prompted to "please explain your answer". Thus it is a remarkable finding – and a further testimony to the power of S1 – that even in such situations, S1 may still "run the show" and lead to non-normative responses, such as the mindless invocation of Lagrange's theorem.

4.2. Educational Implications

On the face of it, the most immediate implication of this work for teaching would be to try to strengthen S2 thinking processes. Our view is different – on two counts. First, strengthening S2 is essentially what mathematics (and science) education has always been trying to do, especially at the university level. As much research in mathematics education and in the H&B literature has shown, this has not been very effective in eliminating biases and misconceptions. Second, our analysis as well as others' indicates that such errors are typically combined failures of both S2 *and* S1, and people often give non-normative answers even when they have the necessary (S2) knowledge to produce correct answers.

Thus we suggest that the most important educational implication from our analysis is the need to train people *to be aware of the way S1 and S2 operate, and to include this awareness in their problem-solving toolbox*. This suggestion has an interesting (almost paradoxical) recursive nature: It in effect implies that S2 should monitor not only the operation of S1 (its standard role), but the S1/S2 interaction as well; thus S2 has to monitor its own functioning in monitoring S1. In a way we might say that an operation of a "System 3" is needed here (to monitor S2), but in practice, this function is recursively assumed by S2 itself. While monitoring and critiquing S1 is one of the reasons S2 has evolved in the first place, monitoring the S1/S2 interaction seems to be what Geary (2002) has called *secondary cognitive ability*, one which will not normally develop without explicit instruction.

As before, this insight has not gone unnoticed in the mathematics education literature, as the following quotation shows:

The main problem is to learn to live with the intuitive loading of concepts – necessary to the productive fluency of reasoning – and, at the same time, to control the impact of the very course of reasoning of these intuitive influences. For this, the student has to learn to become aware of the exact, formal meaning and the implications of the mathematical concepts, on the one hand, and the underlying intuitions on the other. (Fischbein, 1987, p. 207)

More generally, many researchers have stressed the importance for students of reflecting or “looking back” on their solutions (e.g., Polya, 1973; Schoenfeld, 1985, 1987; Schön, 1983, 1987). However, we believe that having students monitor the S1/S2 interaction, and learn to avoid its most common pitfalls, has a degree of specificity – hence trainability – that may be lacking in the older admonitions. If analyzing typical S1/S2 pitfalls became an inherent part of students’ problem solving sessions, they might become more successful problem solvers and decision makers.

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